

VARIANCE OF THE SPECTRAL NUMBERS

CLAUS HERTLING

ABSTRACT. A formula for the variance of the spectrum of a quasihomogeneous singularity is proved, using the G-function of a semisimple Frobenius manifold.

1. INTRODUCTION

The spectrum of an isolated hypersurface singularity $f : (\mathbb{C}^{n+1}, 0) \rightarrow (\mathbb{C}, 0)$ is an important discrete invariant of the singularity. Its main properties have been established by Steenbrink and Varchenko. It consists of μ rational numbers $\alpha_1, \dots, \alpha_\mu$ with $-1 < \alpha_1 \leq \dots \leq \alpha_\mu < n$ and $\alpha_i + \alpha_{\mu+1-i} = n - 1$. The numbers $e^{-2\pi i \alpha_1}, \dots, e^{-2\pi i \alpha_\mu}$ are the eigenvalues of the monodromy.

The spectral numbers come from a Hodge filtration on the cohomology of a Milnor fiber [St1] (cf. chapter 4) or, more instructively, from the Gauß-Manin connection of f [AGV]. They satisfy a semicontinuity property for deformations of f and are related to the signature of the intersection form.

In the case of a quasihomogeneous singularity f of weighted degree 1 with weights $w_0, \dots, w_n \in (0, \frac{1}{2}] \cap \mathbb{Q}$ they can be calculated easily [St2][AGV]: Then the Jacobi algebra $\mathcal{O}_{\mathbb{C}^{n+1}, 0} / (\frac{\partial f}{\partial x_0}, \dots, \frac{\partial f}{\partial x_n}) =: \mathcal{O} / J_f$ has a natural grading $\mathcal{O} / J_f = \bigoplus_\alpha (\mathcal{O} / J_f)_\alpha$, and

$$\sharp(i \mid \alpha_i = \alpha) = \dim(\mathcal{O} / J_f)_{\alpha - \alpha_1} , \quad (1.1)$$

$$\alpha_1 = -1 + \sum_{i=0}^n w_i . \quad (1.2)$$

The main result of this paper is a new formula concerning the distribution of the spectral numbers. Because of the symmetry $\alpha_i + \alpha_{\mu+1-i} = n - 1$ one can consider $\frac{n-1}{2}$ as their *expectation value*. Then $\frac{1}{\mu} \sum_{i=1}^{\mu} (\alpha_i - \frac{n-1}{2})^2$ is their *variance*.

Theorem 1.1. *In the case of a quasihomogeneous singularity the variance is*

$$\frac{1}{\mu} \sum_{i=1}^{\mu} \left(\alpha_i - \frac{n-1}{2} \right)^2 = \frac{\alpha_\mu - \alpha_1}{12} . \quad (1.3)$$

Conjecture 1.2. *For any isolated hypersurface singularity*

$$\frac{1}{\mu} \sum_{i=1}^{\mu} \left(\alpha_i - \frac{n-1}{2} \right)^2 \leq \frac{\alpha_\mu - \alpha_1}{12} . \quad (1.4)$$

The proof of (1.3) uses two things:

1) The deep result of K. Saito [SK1][SK3] and M. Saito [SM] that the base space of a semiuniversal unfolding of an isolated hypersurface singularity f can be equipped with the structure of a Frobenius manifold with discrete invariants related to the spectrum of f .

2) The *G-function* of a semisimple Frobenius manifold, which was defined by Dubrovin and Zhang [DZ1] and independently by Givental [Gi].

A more elementary and much broader version of the construction of K. Saito and M. Saito has been given in [He4] (chapter 6). In chapter 4 we will state the result more precisely. The definition of a Frobenius manifold is given in chapter 3.

The G-function of a semisimple Frobenius manifold is a fascinating function with several origins. One is the τ -*function* of the isomonodromic deformations, which are associated to such a Frobenius manifold. Another is, that in the case of a semisimple Frobenius manifold coming from quantum cohomology the G-function is the genus one Gromov-Witten potential. More remarks on this, the definition, and some properties of the G-function are given in chapter 6.

In order to establish the fact that the G-function extends holomorphically over the *caustic* in the singularity case (Theorem 6.3), we need the *socle field* of a Frobenius manifold (chapter 5) and some facts on *F-manifolds* (chapter 2) from [He3]. If one forgets the metric of a Frobenius manifold one is left with an F-manifold.

Theorem 1.1 is proved in chapter 7.

Conjecture 1.2 is based only on a few examples. I would appreciate a proof as well as counterexamples, also an elementary proof of Theorem 1.1, and applications, for example on deformations of singularities or on their topology.

2. F-MANIFOLDS

First we fix some notations:

1) In the whole paper M is a complex manifold of dimension $m \geq 1$ (with $m = \mu$ in the singularity case) with holomorphic tangent bundle TM , sheaf \mathcal{T}_M of holomorphic vector fields, and sheaf \mathcal{O}_M of holomorphic functions.

2) A (k, l) -tensor is an \mathcal{O}_M -linear map $\mathcal{T}_M^{\otimes k} \rightarrow \mathcal{T}_M^{\otimes l}$. The Lie derivative $\text{Lie}_X T$ of it by a vector field $X \in \mathcal{T}_M$ is again a (k, l) -tensor. For example, a vector field $Y \in \mathcal{T}_M$ yields a $(0, 1)$ -tensor $\mathcal{O}_M \rightarrow \mathcal{T}_M$, $1 \mapsto Y$, with $\text{Lie}_X Y = [X, Y]$.

3) If ∇ is a connection on M then the covariant derivative $\nabla_X T$ of a (k, l) -tensor by a vector field X is again a (k, l) -tensor. As $\nabla_X T$ is \mathcal{O}_M -linear in X (contrary to $\text{Lie}_X T$) ∇T is a $(k+1, l)$ -tensor.

4) A *multiplication* \circ on the tangent bundle TM of a manifold M is a symmetric and associative $(2, 1)$ -tensor $\circ : \mathcal{T}_M \otimes \mathcal{T}_M \rightarrow \mathcal{T}_M$. It equips each tangent space $T_t M$, $t \in M$, with the structure of a commutative and associative \mathbb{C} -algebra. We will be interested only in a multiplication with a global unit field e .

5) A *metric* g on a manifold M is a symmetric and nondegenerate $(2, 0)$ -tensor. It equips each tangent space with a symmetric and nondegenerate bilinear form. Its Levi-Civita connection ∇ is the unique connection on TM

which is torsion free, i.e. $\nabla_X Y - \nabla_Y X = [X, Y]$, and which satisfies $\nabla g = 0$, i.e. $X g(Y, Z) = g(\nabla_X Y, Z) + g(Y, \nabla_X Z)$.

The notion of an F-manifold was defined first in [HM] (cf. [Man] I §5). It is studied extensively in [He3].

Definition 2.1. a) An *F-manifold* (M, \circ, e) is a manifold M together with a multiplication \circ on the tangent bundle and a global unit field e such that the multiplication satisfies the following *integrability condition*:

$$\forall X, Y \in \mathcal{T}_M \quad \text{Lie}_{X \circ Y}(\circ) = X \circ \text{Lie}_Y(\circ) + Y \circ \text{Lie}_X(\circ) . \quad (2.1)$$

b) Let (M, \circ, e) be an F-manifold. An *Euler field of weight* $c \in \mathbb{C}$ is a vector field $E \in \mathcal{T}_M$ with

$$\text{Lie}_E(\circ) = c \cdot \circ . \quad (2.2)$$

An Euler field of weight 1 is simply called an *Euler field*.

One reason why this is a natural and good notion is that any Frobenius manifold is an F-manifold ([HM], [He3] chapter 5). Another one is given in Proposition 2.2 and Theorem 2.3.

Proposition 2.2. ([He3] Prop. 4.1) *The product $(M_1 \times M_2, \circ_1 \oplus \circ_2, e_1 + e_2)$ of two F-manifolds is an F-manifold. The sum (of the lifts to $M_1 \times M_2$) of two Euler fields E_i , $i = 1, 2$, on (M_i, \circ_i, e_i) of the same weight $c \in \mathbb{C}$ is an Euler field of weight $c \in \mathbb{C}$ on $M_1 \times M_2$.*

Theorem 2.3 describes the decomposition of a germ of an F-manifold. In order to state it properly we need the following classical and elementary fact: each tangent space of an F-manifold decomposes as an algebra $(T_t M, \circ, e)$ uniquely into a direct sum

$$(T_t M, \circ, e) = \bigoplus_{k=1}^{l(t)} ((T_t M)_k, \circ, e_k) \quad (2.3)$$

of local subalgebras $(T_t M)_k$ with units e_k and with $(T_t M)_j \circ (T_t M)_k = 0$ for $j \neq k$ (cf. e.g. [He3] Lemma 1.1). One can obtain this decomposition as the simultaneous eigenspace decomposition of the commuting endomorphisms $X \circ : T_t M \rightarrow T_t M$ for $X \in T_t M$.

Theorem 2.3. ([He3] Theorem 4.2) *Let (M, \circ, e) be an F-manifold and $t \in M$. The decomposition (2.3) extends to a unique decomposition*

$$(M, t) = \prod_{k=1}^{l(t)} (M_k, t) \quad (2.4)$$

of the germ (M, t) into a product of germs of F-manifolds. These germs are irreducible germs of F-manifolds as already the algebras $T_t(M_k) \cong (T_t M)_k$ are irreducible (as they are local algebras).

An Euler field of weight c decomposes accordingly.

The proof in [He3] uses (2.1) in a way which justifies calling it *integrability condition*.

Consider an F-manifold (M, \circ, e) . The function $l : M \rightarrow \mathbb{N}$ defined in (2.3) is lower semicontinuous ([He3] Proposition 2.3). The *caustic* $\mathcal{K} := \{t \mid l(t) < \text{generic value}\}$ is empty or a hypersurface (the proof in [He3] Proposition 2.4 for the case *generic value* = m works for any generic value of l).

The multiplication on $T_t M$ is *semisimple* if $l(t) = m$. The F-manifold is *massive* if the multiplication is generically semisimple.

Up to isomorphism there is only one germ of a 1-dimensional F-manifold, (\mathbb{C}, \circ, e) with $e = \frac{\partial}{\partial u}$ for u a coordinate on $(\mathbb{C}, 0)$. The space of Euler fields of weight 0 is $\mathbb{C} \cdot e$, an Euler field of weight 1 is $u e$. This germ of an F-manifold is called A_1 .

By Theorem 2.3, any germ of a semisimple F-manifold is a product A_1^m , that means, there are local coordinates u_1, \dots, u_m with $e_i = \frac{\partial}{\partial u_i}$ and $e_i \circ e_j = \delta_{ij} e_i$. They are unique up to renumbering and shift and are called *canonical coordinates*, following Dubrovin. The vector fields e_i are called *idempotent*. Also by Theorem 2.3, then each Euler field of weight 1 takes the form $\sum_{i=1}^m (u_i + r_i) e_i$ for some $r_i \in \mathbb{C}$.

Example 2.4. Fix $m \geq 1$ and $n \geq 2$. The manifold $M = \mathbb{C}^m$ with coordinate fields $\delta_i = \frac{\partial}{\partial t_i}$ and multiplication defined by

$$\delta_1 \circ \delta_2 = \delta_2, \quad (2.5)$$

$$\delta_2 \circ \delta_2 = t_2^{n-2} \delta_1, \quad (2.6)$$

$$\delta_i \circ \delta_j = \delta_{ij} \delta_i \quad \text{if } (i, j) \notin \{(1, 2), (2, 1), (2, 2)\} \quad (2.7)$$

is a massive F-manifold. The submanifold $\mathbb{C}^2 \times \{0\}$ is an F-manifold with the name $I_2(n)$, with $I_2(2) = A_1^2$, $I_2(3) = A_2$, $I_2(4) = B_2$, $I_2(5) = H_2$, and $I_2(6) = G_2$.

(M, \circ, e) decomposes globally into a product $\mathbb{C}^2 \times \mathbb{C} \times \dots \times \mathbb{C}$ of F-manifolds of the type $I_2(n) A_1^{m-2}$. The unit fields for the components are $\delta_1, \delta_3, \dots, \delta_m$, the global unit field is $e = \delta_1 + \delta_3 + \dots + \delta_m$, the caustic is $\mathcal{K} = \{t \mid t_2 = 0\}$. The idempotent vector fields in a simply connected subset of $M - \mathcal{K}$ are

$$e_{1/2} = \frac{1}{2} \delta_1 \pm \frac{1}{2} t_2^{-\frac{n-2}{2}} \delta_2, \quad (2.8)$$

$$e_i = \delta_i \quad \text{for } i \geq 3, \quad (2.9)$$

canonical coordinates there are

$$u_{1/2} = t_1 \pm \frac{2}{n} t_2^{\frac{n}{2}}, \quad (2.10)$$

$$u_i = t_i \quad \text{for } i \geq 3. \quad (2.11)$$

An Euler field of weight 1 is

$$E = t_1 \delta_1 + \frac{2}{m} t_2 \delta_2 + \sum_{i \geq 3} t_i \delta_i. \quad (2.12)$$

The space of global Euler fields of weight 0 is $\sum_{i \neq 2} \mathbb{C} \cdot \delta_i$.

The classification of 3-dimensional irreducible germs of massive F-manifolds is already vast ([He3] chapter 20). But the classification of 2-dimensional irreducible germs of massive F-manifolds is nice ([He3] Theorem 12.1): they are precisely the germs at 0 for $m = 2$ and $n \geq 3$ in Example 2.4 with the names $I_2(n)$.

3. FROBENIUS MANIFOLDS

Frobenius manifolds were defined first by Dubrovin [Du1]. They turn up now at many places, see [Du2] and [Man] (also for more references), especially in quantum cohomology and mirror symmetry. In this paper we will only be concerned with the Frobenius manifolds in singularity theory (chapter 4).

Definition 3.1. A Frobenius manifold (M, \circ, e, E, g) is a manifold M with a multiplication \circ on the tangent bundle, a global unit field e , another global vector field E , which is called *Euler field*, and a metric g , subject to the following conditions:

- 1) the metric is *multiplication invariant*, $g(X \circ Y, Z) = g(X, Y \circ Z)$,
- 2) (potentiality) the $(3, 1)$ -tensor $\nabla \circ$ is symmetric (here ∇ is the Levi-Civita connection of the metric),
- 3) the metric g is flat,
- 4) the unit field e is flat, $\nabla e = 0$,
- 5) the Euler field satisfies $\text{Lie}_E(\circ) = 1 \cdot \circ$ and $\text{Lie}_E(g) = D \cdot g$ for some $D \in \mathbb{C}$.

Remarks 3.2. a) Condition 2) implies (2.1) ([He3] Theorem 5.2). Therefore a Frobenius manifold is an F-manifold.

b) The $(3, 0)$ -tensor A with $A(X, Y, Z) := g(X \circ Y, Z)$ is symmetric by 1). Then 2) is equivalent to the symmetry of the $(4, 0)$ -tensor ∇A . If X, Y, Z, W are local flat fields then 2) is equivalent to the symmetry in X, Y, Z, W of $X A(Y, Z, W)$. This is equivalent to the existence of a local *potential* $\Phi \in \mathcal{O}_M$ with $A(X, Y, Z) = XYZ(\Phi)$ for flat fields X, Y, Z .

c) $\text{Lie}_E(g) = D \cdot g$ means that E is a sum of an infinitesimal dilation, rotation and shift. Therefore ∇E maps a flat field X to a flat field $\nabla_X E = [X, E] = -\text{Lie}_E X$, i.e. it is a flat $(1, 1)$ -tensor, $\nabla(\nabla E) = 0$. Its eigenvalues are called d_1, \dots, d_m . Now $\text{Lie}_E(\circ) = 1 \cdot \circ$ implies $\nabla_e E = [e, E] = e$, and $\nabla E - \frac{D}{2} \text{id}$ is an infinitesimal isometry because of $\text{Lie}_E(g) = D \cdot g$. One can order the eigenvalues such that $d_1 = 1$ and $d_i + d_{m+1-i} = D$.

4. HYPERSURFACE SINGULARITIES

Let $f : (\mathbb{C}^{n+1}, 0) \rightarrow (\mathbb{C}, 0)$ be a holomorphic function germ with an isolated singularity at 0 and with Milnor number μ .

An unfolding of f is a holomorphic function germ $F : (\mathbb{C}^{n+1} \times \mathbb{C}^m, 0) \rightarrow (\mathbb{C}, 0)$ with $F|_{(\mathbb{C}^{n+1} \times \{0\}, 0)} = f$. The coordinates on $(\mathbb{C}^{n+1} \times \mathbb{C}^m, 0)$ are called $(x_0, \dots, x_n, t_1, \dots, t_m)$.

The germ $(C, 0) \subset (\mathbb{C}^{n+1} \times \mathbb{C}^m, 0)$ of the critical space is defined by the ideal $J_F = (\frac{\partial F}{\partial x_0}, \dots, \frac{\partial F}{\partial x_n})$. The projection $pr_C : (C, 0) \rightarrow (\mathbb{C}^m, 0)$ is finite and flat of

degree μ . The map

$$\mathbf{a} : \mathcal{T}_{\mathbb{C}^m, 0} \longrightarrow \mathcal{O}_{C, 0} \quad (4.1)$$

$$\frac{\partial}{\partial t_i} \longmapsto \frac{\partial F}{\partial t_i}|_{(C, 0)} \quad (4.2)$$

is the Kodaira-Spencer map.

The unfolding is semiuniversal iff \mathbf{a} is an isomorphism. Consider a semiuniversal unfolding. Then $m = \mu$, and we set $(M, 0) := (\mathbb{C}^\mu, 0)$. The map \mathbf{a} induces a multiplication \circ on $\mathcal{T}_{M, 0}$ by $\mathbf{a}(X \circ Y) = \mathbf{a}(X) \cdot \mathbf{a}(Y)$ with unit field $e = \mathbf{a}^{-1}([1])$ and a vector field $E := \mathbf{a}^{-1}([F])$.

This multiplication and the field E were considered first by K. Saito [SK1][SK3]. Because the Kodaira-Spencer map \mathbf{a} behaves well under morphisms of unfoldings, the tuple $((M, 0), \circ, e, E)$ is essentially independent of the choices of the semiuniversal unfolding. This and the following fact are discussed in [He3] (chapter 16).

Theorem 4.1. *The base space $(M, 0)$ of a semiuniversal unfolding F of an isolated hypersurface singularity f is a germ $((M, 0), \circ, e, E)$ of a massive F -manifold with Euler field $E = \mathbf{a}^{-1}([F])$. For each $t \in M$ there is a canonical isomorphism*

$$(T_t M, \circ, E|_t) \cong \left(\bigoplus_{x \in \text{Sing}(F_t)} \text{Jacobi algebra of } (F_t, x), \text{mult.}, [F_t] \right). \quad (4.3)$$

At generic points of the caustic the germ of the F -manifold is of the type $A_2 A_1^{\mu-2}$.

The base spaces of two semiuniversal unfoldings are canonically isomorphic as germs of F -manifolds with Euler fields.

By work of K. Saito [SK1][SK3] and M. Saito [SM] one can even construct a metric g on M such that $((M, 0), \circ, e, E, g)$ is the germ of a Frobenius manifold. The construction uses the Gauß-Manin connections for f and F , K. Saito's higher residue pairings, a polarized mixed Hodge structure, and results of Malgrange on deformations of microdifferential systems.

A more elementary and much broader version of the construction, which does not use Malgrange's results, is given in [He4]. Here we restrict ourselves to a formulation of the result. This uses the *spectral numbers* of f [St1][AGV].

Let $f : X \rightarrow \Delta$ be a representative of the germ f as usual, with $\Delta = B_\delta \subset \mathbb{C}$ and $X = f^{-1}(\Delta) \cap B_\varepsilon^{n+1} \subset \mathbb{C}^{n+1}$ (with $1 \gg \varepsilon \gg \delta > 0$). The cohomology bundle $H^n = \bigcup_{t \in \Delta^*} H^n(f^{-1}(t), \mathbb{C})$ is flat. Denote by $\Delta^\infty \rightarrow \Delta^*$ a universal covering. A *global flat multivalued section* in H^n is a map $\Delta^\infty \rightarrow H^n$ with the obvious properties. The μ -dimensional space of the global flat multivalued sections in H^n is denoted by H^∞ . Steenbrink's Hodge filtration F^\bullet on H^∞ [St1] together with topological data yields a polarized mixed Hodge structure on it (see [He1][He4] for definitions and a discussion of this).

The *spectral numbers* $\alpha_1, \dots, \alpha_\mu$ of f are μ rational numbers with

$$\sharp(i \mid \alpha_i = \alpha) = \dim \text{Gr}_F^{[n-\alpha]} H_{e^{-2\pi i \alpha}}^\infty. \quad (4.4)$$

Here $H_{e^{-2\pi i\alpha}}^\infty$ is the generalized eigenspace of the monodromy on H^∞ with eigenvalue $e^{-2\pi i\alpha}$. So $e^{-2\pi i\alpha_1}, \dots, e^{-2\pi i\alpha_\mu}$ are the eigenvalues of the monodromy. The spectral numbers satisfy $-1 < \alpha_1 \leq \dots \leq \alpha_\mu < n$ and $\alpha_i + \alpha_{\mu+1-i} = n - 1$.

Essential for understanding them and for the whole construction of the metric g is Varchenko's way to construct a mixed Hodge structure on H^∞ with the Gauß-Manin connection of f ([Va][AGV], cf. also [SchSt][SM][He1][He4]).

It turns out that a metric g such that $((M, 0), \circ, e, E, g)$ is a Frobenius manifold is in general not unique. By work of K. Saito and M. Saito each choice of a filtration on H^∞ which is *opposite* to F^\bullet (see [SM][He4] for the definition) yields a metric which gives a Frobenius manifold structure as in Theorem 4.2. A more precise statement and a detailed proof of it is given in [He4] (chapter 6).

Theorem 4.2. *One can choose a metric g on the base space $(M, 0)$ of a semiuniversal unfolding of an isolated hypersurface singularity f such that $((M, 0), \circ, e, E, g)$ is a germ of a Frobenius manifold and ∇E is semisimple with eigenvalues $d_i = 1 + \alpha_1 - \alpha_i$ and with $D = 2 - (\alpha_\mu - \alpha_1)$.*

In fact, often one can also find metrics giving Frobenius manifold structures with $\{d_1, \dots, d_\mu\} \neq \{1 + \alpha_1 - \alpha_i \mid i = 1, \dots, \mu\}$ ([SM] 4.4, [He4] Remarks 6.7).

5. SOCLE FIELD

A Frobenius manifold has another distinguished vector field besides the unit field and the Euler field. It will be discussed in this section. We call it the *socle field*. It is used implicitly in Dubrovin's papers and in [Gi].

Let (M, \circ, e, g) be a manifold with a multiplication \circ on the tangent bundle, with a unit field, and with a multiplication invariant metric g . We do not need flatness and potentiality and an Euler field in the moment.

Each tangent space $T_t M$ is a Frobenius algebra. This means (more or less by definition) that the splitting (2.3) is now a splitting into a direct sum of Gorenstein rings (cf. e.g. [He3] Lemma 1.2)

$$T_t M = \bigoplus_{k=1}^{l(t)} (T_t M)_k \quad (5.1)$$

They have maximal ideals $\mathbf{m}_{t,k} \subset (T_t M)_k$ and units e_k such that $e = \sum e_k$. They satisfy

$$(T_t M)_j \circ (T_t M)_k = \{0\} \quad \text{for } j \neq k, \quad (5.2)$$

and thus

$$g((T_t M)_j, (T_t M)_k) = \{0\} \quad \text{for } j \neq k. \quad (5.3)$$

The socle $\text{Ann}_{(T_t M)_k}(\mathbf{m}_{t,k})$ is 1-dimensional and has a unique generator $H_{t,k}$ which is normalized such that

$$g(e_k, H_{t,k}) = \dim(T_t M)_k. \quad (5.4)$$

The following lemma shows that the vectors $\sum_k H_{t,k}$ glue to a holomorphic vector field, the *socle field* of (M, \circ, e, g) .

Lemma 5.1. *For any dual bases X_1, \dots, X_m and $\tilde{X}_1, \dots, \tilde{X}_m$ of $T_t M$, that means, $g(X_i, \tilde{X}_j) = \delta_{ij}$, one has*

$$\sum_{k=1}^{l(t)} H_{t,k} = \sum_{i=1}^m X_i \circ \tilde{X}_i . \quad (5.5)$$

Proof: One sees easily that the sum $\sum X_i \circ \tilde{X}_i$ is independent of the choice of the basis X_1, \dots, X_m . One can suppose that $l(t) = 1$ and that X_1, \dots, X_m are chosen such that they yield a splitting of the filtration $T_t M \supset \mathfrak{m}_{t,1} \supset \mathfrak{m}_{t,1}^2 \supset \dots$. Then $g(e, X_i \circ \tilde{X}_i) = 1$ and

$$g(\mathfrak{m}_{t,1}, X_i \circ \tilde{X}_i) = g(X_i \circ \mathfrak{m}_{t,1}, \tilde{X}_i) = 0 .$$

Thus $X_i \circ \tilde{X}_i = \frac{1}{m} H_{t,1}$. \square

It will be useful to fix the multiplication and vary the metric.

Lemma 5.2. *Let (M, \circ, e, g) be a manifold with multiplication \circ on the tangent bundle, unit field e and multiplication invariant metric g . For each multiplication invariant metric \tilde{g} there exists a unique vector field Z such that the multiplication with it is invertible everywhere and for all vector fields X, Y*

$$\tilde{g}(X, Y) = g(Z \circ X, Y) . \quad (5.6)$$

The socle fields H and \tilde{H} of g and \tilde{g} satisfy

$$H = Z \circ \tilde{H} . \quad (5.7)$$

Proof: The situation for one Frobenius algebra is described for example in [He3] (Lemma 1.2). It yields (5.6) immediately. (5.7) follows from the comparison of (5.4) and (5.6). \square

Denote by

$$H_{op} : \mathcal{T}_M \rightarrow \mathcal{T}_M, \quad X \mapsto H \circ X \quad (5.8)$$

the multiplication with the socle field H of (M, \circ, e, g) as above. The socle field is especially interesting if the multiplication is generically semisimple, that means, generically $l(t) = m$. Then the caustic $\mathcal{K} = \{t \in M \mid l(t) < m\}$ is the set where the multiplication is not semisimple. It is the hypersurface

$$\mathcal{K} = \det(H_{op})^{-1}(0) . \quad (5.9)$$

In an open subset of $M - \mathcal{K}$ with basis e_1, \dots, e_m of idempotent vector fields the socle field is

$$H = \sum_{i=1}^m \frac{1}{g(e_i, e_i)} e_i . \quad (5.10)$$

It determines the metric g everywhere because (5.10) determines the metric at semisimple points.

Theorem 5.3. *Let (M, \circ, e, g) be a massive F -manifold with multiplication invariant metric g . Suppose that at generic points of the caustic the germ of the F -manifold is of the type $I_2(n)A_1^{m-2}$.*

Then the function $\det(H_{op})$ vanishes with multiplicity $n-2$ along the caustic.

Proof: It is sufficient to consider Example 2.4. A multiplication invariant metric g is uniquely determined by the 1-form $\varepsilon = g(e, \cdot)$. Because of (5.7) it is sufficient to prove the claim for one metric. We choose the metric with 1-form

$$\varepsilon(\delta_i) = 1 - \delta_{i1} . \quad (5.11)$$

The bases $\delta_1, \delta_2, \delta_3, \dots, \delta_m$ and $\delta_2, \delta_1, \delta_3, \dots, \delta_m$ are dual with respect to this metric. Its socle field is by Lemma 5.1

$$H = 2\delta_2 + \delta_3 + \dots + \delta_m \quad (5.12)$$

and satisfies $\det(H_{op}) = -4t_2^{n-2}$. \square

6. G-FUNCTION OF A MASSIVE FROBENIUS MANIFOLD

Associated to any simply connected semisimple Frobenius manifold is a fascinating and quite mysterious function. Dubrovin and Zhang [DZ1][DZ2] called it the G-function and proved the deepest results on it. But Givental [Gi] studied it, too, and it originates in much older work. It takes the form

$$G(t) = \log \tau_I - \frac{1}{24} \log J \quad (6.1)$$

and is determined only up to addition of a constant. First we explain the simpler part, $\log J$. Let (M, \circ, e, E, g) be a semisimple Frobenius manifold with canonical coordinates u_1, \dots, u_m and flat coordinates $\tilde{t}_1, \dots, \tilde{t}_m$. Then

$$J = \det\left(\frac{\partial \tilde{t}_i}{\partial u_j}\right) \cdot \text{constant} \quad (6.2)$$

is the base change matrix between flat and idempotent vector fields.

One can rewrite it with the socle field. Denote $\eta_i := g(e_i, e_i)$ and consider the basis v_1, \dots, v_m of vector fields with

$$v_i = \frac{1}{\sqrt{\eta_i}} e_i \quad (6.3)$$

(for some choice of the square roots). The matrix $\det(g(v_i, v_j)) = 1$ is constant as is the corresponding matrix for the flat vector fields. Therefore

$$\text{constant} \cdot J = \prod_{i=1}^m \sqrt{\eta_i} = \det(H_{op})^{-\frac{1}{2}} . \quad (6.4)$$

Here $H = \sum v_i \circ v_i$ is the socle field.

One of the origins of the first part $\log \tau_I$ are isomonodromic deformations. The second structure connections and the first structure connections of the semisimple Frobenius manifold are isomonodromic deformations over $\mathbb{P}^1 \times M$ of restrictions to a slice $\mathbb{P}^1 \times \{t\}$. The function τ_I is their τ -function in the sense of [JMMS][JMU][JM][Mal]. See [Sab] for other general references on this.

The situation for Frobenius manifolds is discussed and put into a Hamiltonian framework in [Du2] (Lecture 3), [Man] (II §2), and in [Hi]. The coefficients H_i of the 1-form $d \log \tau_I = \sum H_i du_i$ are certain Hamiltonians and motivate the definition of this 1-form. Hitchin [Hi] compares the realizations of this for the first and the second structure connections.

Another origin of the whole G-function comes from quantum cohomology. Getzler [Ge] studied the relations between cycles in the moduli space $\overline{\mathcal{M}}_{1,4}$ and derived from it recursion relations for genus one Gromov-Witten invariants of projective manifolds and differential equations for the genus one Gromov-Witten potential.

Dubrovin and Zhang [DZ1] (chapter 6) investigated these differential equations for any semisimple Frobenius manifold and found that they have always one unique solution (up to addition of a constant), the G-function. Therefore in the case of a semisimple Frobenius manifold coming from quantum cohomology, the G-function is the genus one Gromov-Witten potential. (Still I find these differential equations mysterious.)

They also proved part of the conjectures in [Gi] concerning $G(t)$. Finally, they found that the potential of the Frobenius manifold (for genus zero) and the G-function (for genus one) are the basements of full free energies in genus zero and one and give rise to Virasoro constraints [DZ2]. Exploiting this for singularities will be a big task for the future.

In chapter 7 we need only the definition of $\log \tau_I$ and the behaviour of $G(t)$ with respect to the Euler field and the caustic in a massive Frobenius manifold. We have to resume some known formulas related to the canonical coordinates of a semisimple Frobenius manifold ([Du2], [Man], also [Gi]).

The 1-form $\varepsilon = g(e, \cdot)$ is closed and can be written as $\varepsilon = d\eta$. One defines

$$\eta_i := e_i \eta = g(e_i, e) = g(e_i, e_i) , \quad (6.5)$$

$$\eta_{ij} := e_i e_j \eta = e_i \eta_j = e_j \eta_i , \quad (6.6)$$

$$\gamma_{ij} := \frac{1}{2} \frac{\eta_{ij}}{\sqrt{\eta_i} \sqrt{\eta_j}} , \quad (6.7)$$

$$V_{ij} := -(u_i - u_j) \gamma_{ij} , \quad (6.8)$$

$$d \log \tau_I := \frac{1}{2} \sum_{i \neq j} \frac{V_{ij}^2}{u_i - u_j} du_i = \frac{1}{2} \sum_{i \neq j} (u_i - u_j) \gamma_{ij}^2 du_i \quad (6.9)$$

$$= \frac{1}{8} \sum_{i \neq j} (u_i - u_j) \frac{\eta_{ij}^2}{\eta_i \eta_j} du_i . \quad (6.10)$$

Theorem 6.1. *Let (M, \circ, e, E, g) be a semisimple Frobenius manifold with global canonical coordinates u_1, \dots, u_m .*

a) The rotation coefficients γ_{ij} (for $i \neq j$) satisfy the Darboux-Egoroff equations

$$e_k \gamma_{ij} = \gamma_{ik} \gamma_{kj} \quad \text{for } k \neq i \neq j \neq k , \quad (6.11)$$

$$e \gamma_{ij} = 0 \quad \text{for } i \neq j . \quad (6.12)$$

b) The connection matrix of the flat connection for the basis v_1, \dots, v_m from (6.3) is the matrix $\Gamma := (\gamma_{ij}d(u_i - u_j))$. The Darboux-Egoroff equations are equivalent to the flatness condition $d\Gamma + \Gamma \wedge \Gamma = 0$.

c) The 1-form $d \log \tau_I$ is closed and comes from a function $\log \tau_I$.

d) $E(\eta_i) = (D - 2)\eta_i$ and $E(\gamma_{ij}) = -\gamma_{ij}$.

e) If the canonical coordinates are chosen such that $E = \sum u_i e_i$ then the matrix $-(V_{ij})$ is the matrix of the endomorphism $\mathcal{V} = \nabla E - \frac{D}{2} \text{id}$ on \mathcal{T}_M with respect to the basis v_1, \dots, v_m .

Proof: a)+b) See [Du2] (pp. 200–201) or [Man] (I §3).

c) This can be checked easily with the Darboux-Egoroff equations.

d) It follows from $\text{Lie}_E(g) = D \cdot g$ and from $[e_i, E] = e_i$.

e) This is implicit in [Du2] (pp. 200–201). One can check it with a)+b)+d). \square

The endomorphism \mathcal{V} is skewsymmetric with respect to g and flat with eigenvalues $d_i - \frac{D}{2}$; the numbers d_i can be ordered such that $d_1 = 1$ and $d_i + d_{m+1-i} = D$ (cf. Remark 3.2 c)).

Corollary 6.2. ([DZ1] Theorem 3) Suppose that $E = \sum u_i e_i$. Then

$$E \log \tau_I = -\frac{1}{4} \sum_{i=1}^m (d_i - \frac{D}{2})^2, \quad (6.13)$$

$$E G(t) = -\frac{1}{4} \sum_{i=1}^m (d_i - \frac{D}{2})^2 + \frac{m(2-D)}{48} =: \gamma. \quad (6.14)$$

Proof:

$$\begin{aligned} E \log \tau_I &= \frac{1}{2} \sum_{i \neq j} \frac{u_i V_{ij}^2}{u_i - u_j} = \frac{1}{2} \sum_{i < j} V_{ij}^2 \\ &= -\frac{1}{4} \sum_{ij} V_{ij} V_{ji} = -\frac{1}{4} \text{trace}(V^2) \\ &= -\frac{1}{4} \sum_{i=1}^m (d_i - \frac{D}{2})^2. \end{aligned} \quad (6.15)$$

(6.4) shows $E(J) = m \frac{D-2}{2} J$. Now (6.14) follows from the definition of the G-function. \square

If M is a massive Frobenius manifold with caustic \mathcal{K} , one may ask which kind of poles the 1-form $d \log \tau_I$ has along \mathcal{K} and when the G-function extends over \mathcal{K} .

All the F-manifolds $I_2(n)$ (cf. Example 2.4) are in a natural way (up to the choice of a scalar) equipped with a metric g , such that they get Frobenius manifolds ([Du2] Lecture 4, cf. [He3] chapter 19). These Frobenius manifolds are also denoted $I_2(n)$.

In [DZ1] (chapter 6) the G-function is calculated for them with coordinates (t_1, t_2) on $M = \mathbb{C}^2$ and $e = \frac{\partial}{\partial t_1}$. It turns out to be

$$G(t) = -\frac{1}{24} \frac{(2-n)(3-n)}{n} \log t_2 . \quad (6.16)$$

Especially, for the case $I_2(3) = A_2$ the G-function is $G(t) = 0$. This was checked independently in [Gi]. Givental concluded that in the case of singularities the G-function of the base space of a semiuniversal unfolding with some Frobenius manifold structure extends holomorphically over the caustic. This is a good guess, but it does not follow from the case A_2 , because a Frobenius manifold structure on a germ of an F-manifold of type $A_2 A_1^{m-2}$ for $m \geq 3$ is never the product of the Frobenius manifolds A_2 and A_1^{m-2} (the numbers d_1, \dots, d_m would not be symmetric). Anyway, it is true, as the following result shows.

Theorem 6.3. *Let (M, \circ, e, E, g) be a simply connected massive Frobenius manifold. Suppose that at generic points of the caustic \mathcal{K} the germ of the underlying F-manifold is of type $I_2(n) A_1^{m-2}$ for one fixed number $n \geq 3$.*

a) *The form $d \log \tau_I$ has a logarithmic pole along \mathcal{K} with residue $-\frac{(n-2)^2}{16n}$ along \mathcal{K}_{reg} .*

b) *The G-function extends holomorphically over \mathcal{K} iff $n = 3$.*

Proof: Theorem 5.3 and (6.4) say that the form $-\frac{1}{24} d \log J$ has a logarithmic pole along \mathcal{K} with residue $\frac{n-2}{48}$ along \mathcal{K}_{reg} . This equals $\frac{(n-2)^2}{16n}$ iff $n = 3$. So b) follows from a).

It is sufficient to show a) for the F-manifold in Example 2.4, equipped with some metric which makes a Frobenius manifold out of it (we do not need an Euler field here). Unfortunately we do not have an identity for $d \log \tau_I$ as (5.7) for the socle field which would allow to consider only a most convenient metric.

We use (2.5) - (2.11) and (6.5) - (6.10) and consider a neighborhood of $0 \in \mathbb{C}^m = M$. Denote for $j \geq 3$

$$\begin{aligned} T_{1j} &:= (u_1 - u_j) \frac{\eta_{j1}^2}{\eta_j \eta_1} + (u_2 - u_j) \frac{\eta_{j2}^2}{\eta_j \eta_2} , \\ T_{2j} &:= (u_1 - u_j) \frac{\eta_{j1}^2}{\eta_j \eta_1} - (u_2 - u_j) \frac{\eta_{j2}^2}{\eta_j \eta_2} , \\ T_{12} &:= (u_1 - u_2) \frac{\eta_{12}^2}{\eta_1 \eta_2} d(u_1 - u_2) . \end{aligned} \quad (6.17)$$

With $\eta_j(0) \neq 0$ for $j \geq 3$, (6.10) and (2.10) one calculates

$$\begin{aligned} 8d \log \tau_I &= \text{holomorphic 1-form} + T_{12} \\ &+ \sum_{j \geq 3} T_{1j} dt_1 + \sum_{j \geq 3} T_{2j} t_2^{\frac{n-2}{2}} dt_2 \\ &- \sum_{j \geq 3} T_{1j} du_j . \end{aligned} \quad (6.18)$$

From (2.8) one obtains

$$\eta_{1/2} = \frac{1}{2}\delta_1(\eta) \pm \frac{1}{2}\delta_2(\eta)t_2^{-\frac{n-2}{2}}, \quad (6.19)$$

$$\eta_1 \cdot \eta_2 = \frac{1}{4}t_2^{-n+2}(-\delta_2(\eta)^2 + t_2^{n-2}\delta_1(\eta)^2), \quad (6.20)$$

$$\eta_{12} = \frac{1}{4}\delta_1\delta_2(\eta) + \frac{1}{4}\frac{n-2}{2}t_2^{-n+1}\delta_2(\eta) - \frac{1}{4}t_2^{-n+2}\delta_2\delta_2(\eta). \quad (6.21)$$

The vector $\delta_2|_0$ is a generator of the socle of the subalgebra in T_0M which corresponds to $I_2(n)$. Therefore $\delta_2(\eta)(0) \neq 0$. It is not hard to see with (6.19) and (2.10) that the terms T_{1j} and $T_{2j}t_2^{\frac{n-2}{2}}$ for $j \geq 3$ are holomorphic at 0. The term T_{12} is

$$\begin{aligned} T_{12} &= \frac{4}{n} \cdot t_2^{\frac{n}{2}} \cdot \frac{\eta_{12}^2}{\eta_1\eta_2} \cdot d\left(\frac{4}{n} \cdot t_2^{\frac{n}{2}}\right) \\ &= \frac{8}{n} \cdot t_2^{n-1} \cdot \frac{\eta_{12}^2}{\eta_1\eta_2} \cdot dt_2 \\ &= -\frac{(n-2)^2}{2n} \cdot \frac{dt_2}{t_2} + \text{holomorphic 1-form}. \end{aligned} \quad (6.22)$$

This proves part a). \square

Remark 6.4. It might be interesting to look for massive Frobenius manifolds which meet the case $n = 3$ in Theorem 6.3, but where the underlying F-manifolds are not locally products of those from hypersurface singularities. In view of [He3] (Theorem 16.6) the *analytic spectrum* $\text{Specan}(\mathcal{T}_M, \circ) \subset T^*M$ of such F-manifolds would have singularities, but only in codimension ≥ 2 , as the analytic spectrum of A_2 is smooth.

The analytic spectrum is Cohen-Macaulay and even Gorenstein and a Lagrange variety ([He3] chapter 6). P. Seidel (Ecole Polytechnique) showed me a normal and Cohen-Macaulay Lagrange surface. But it seems to be unclear whether there exist normal and Gorenstein Lagrange varieties which are not smooth.

7. VARIANCE OF THE SPECTRUM

By Theorem 6.3 the germ $(M, 0)$ of a Frobenius manifold as in Theorem 4.2 for an isolated hypersurface singularity f has a holomorphic G-function $G(t)$, unique up to addition of a constant. By Corollary 6.2 and Theorem 4.2 this function satisfies

$$E G(t) = -\frac{1}{4} \sum_{i=1}^{\mu} \left(\alpha_i - \frac{n-1}{2}\right)^2 + \frac{\mu(\alpha_{\mu} - \alpha_1)}{48} =: \gamma. \quad (7.1)$$

So it has a very peculiar strength: it gives a grip at the squares of the spectral numbers $\alpha_1, \dots, \alpha_{\mu}$ of the singularity. Because of the symmetry $\alpha_i + \alpha_{\mu+1-i} = n-1$, the spectral numbers are scattered around their *expectation value* $\frac{n-1}{2}$. One may ask about their *variance* $\frac{1}{\mu} \sum_{i=1}^{\mu} (\alpha_i - \frac{n-1}{2})^2$.

Conjecture 7.1. *The variance of the spectral numbers of an isolated hyper-surface singularity is*

$$\frac{1}{\mu} \sum_{i=1}^{\mu} (\alpha_i - \frac{n-1}{2})^2 \leq \frac{\alpha_{\mu} - \alpha_1}{12} , \quad (7.2)$$

or, equivalently,

$$\gamma \geq 0 . \quad (7.3)$$

Theorem 7.2. *In the case of a quasihomogeneous singularity f*

$$\frac{1}{\mu} \sum_{i=1}^{\mu} (\alpha_i - \frac{n-1}{2})^2 = \frac{\alpha_{\mu} - \alpha_1}{12} , \quad (7.4)$$

and

$$\gamma = 0 . \quad (7.5)$$

Proof: $(\mathcal{O}/J_f, \text{mult.}, [f]) \cong (T_0 M, \circ, E|_0)$. Here one has $f \in J_f$ and $E|_0 = 0$ and therefore $E G(t) = 0$. \square

Lemma 7.3. *The number γ of the sum $f(x_0, \dots, x_n) + g(y_0, \dots, y_m)$ of two singularities f and g satisfies*

$$\gamma(f + g) = \mu(f) \cdot \gamma(g) + \mu(g) \cdot \gamma(f) . \quad (7.6)$$

Proof: Let $\alpha_1, \dots, \alpha_{\mu(f)}$ and $\beta_1, \dots, \beta_{\mu(g)}$ denote the spectral numbers of f and g . Then the spectrum of $f + g$ as an unordered tuple is [AGV][SchSt]

$$(\alpha_i + \beta_j + 1 \mid i = 1, \dots, \mu(f), j = 1, \dots, \mu(g)) . \quad (7.7)$$

This and the symmetry of the spectra yields (7.6). \square

Remarks 7.4. a) The only unimodal or bimodal families of not semiquasihomogeneous singularities are the cusp singularities T_{pqr} and the 8 bimodal series. The spectral numbers are given in [AGV]. One finds

$$\gamma(T_{pqr}) = \frac{1}{24} (1 - \frac{1}{p} - \frac{1}{q} - \frac{1}{r}) \geq 0 \quad (7.8)$$

with equality only for the simple elliptic singularities. In the case of the 8 bimodal families one obtains

$$\gamma = \frac{p}{48 \cdot \kappa} \cdot \left(1 - \frac{1}{p + \kappa} \right) \geq 0 \quad (7.9)$$

with $\kappa := 9, 7, 6, 6, 5$ for $E_{3,p}, Z_{1,p}, Q_{2,p}, W_{1,p}, S_{1,p}$, respectively, and

$$\gamma = \frac{p}{48 \cdot \kappa} \cdot \left(1 + \frac{1}{p + 2\kappa} \right) \geq 0 \quad (7.10)$$

with $\kappa := 6, 5, \frac{9}{2}$ for $W_{1,p}^{\sharp}, S_{1,p}^{\sharp}, U_{1,p}$, respectively.

b) Checking $\gamma = 0$ for the A_{μ} -singularities is easy. With Lemma 7.3 one obtains immediately $\gamma = 0$ for all Brieskorn-Pham singularities. But this is far from a general elementary proof of Theorem 7.2 for all quasihomogeneous singularities.

c) In [SK2] K. Saito studied the distribution of the spectral numbers and their characteristic function

$$\chi_f := \frac{1}{\mu} \sum_{i=1}^{\mu} T^{\alpha_i+1} \quad (7.11)$$

heuristically and formulated several questions about them. The G-function might help to go on with these problems.

d) In the case of a quasihomogeneous singularity with weights $w_0, \dots, w_n \in (0, \frac{1}{2}]$ and degree 1 the characteristic function is

$$\chi_f = \frac{1}{\mu} \prod_{i=0}^n \frac{T - T^{w_i}}{T^{w_i} - 1}, \quad (7.12)$$

as is well known. It follows easily from (1.1) and (1.2).

e) One can speculate that the Conjecture 7.1, if it is true, comes from a deeper hidden interrelation between the Gauß-Manin connection and polarized mixed Hodge structures.

In [He4] (Remark 6.7 b)) an example of M. Saito ([SM] 4.4) is sketched which leads for the semiquasihomogeneous singularity $f = x^6 + y^6 + x^4 y^4$ to Frobenius manifold structures with $\{d_1, \dots, d_\mu\} \neq \{1 + \alpha_1 - \alpha_i \mid i = 1, \dots, \mu\}$. The number γ in that case is $\gamma = -\frac{1}{144} < 0$.

f) In the case of the simple singularities A_k, D_k, E_6, E_7, E_8 , the parameters t_1, \dots, t_μ of a suitably chosen unfolding are weighted homogeneous with positive degrees with respect to the Euler field. Therefore $G = 0$ in these cases (cf. [Gi]).

REFERENCES

- [AGV] Arnold, V.I., S.M. Gusein-Zade, A.N. Varchenko: Singularities of differentiable maps, volume II. Birkhäuser, Boston 1988.
- [Du1] Dubrovin, B.: Integrable systems in topological field theory. Nucl. Phys. **B 375** (1992), 627–685.
- [Du2] Dubrovin, B.: Geometry of 2D topological field theories. In: Integrable systems and quantum groups. Montecatini, Terme 1993 (M. Francoviglia, S. Greco, eds.). Lecture Notes in Math. 1620, Springer-Verlag 1996, pp. 120–348.
- [DZ1] Dubrovin, B., Y. Zhang: Bihamiltonian hierarchies in 2D topological field theory at one-loop approximation. Commun. Math. Phys. **198** (1998), 311–361.
- [DZ2] Dubrovin, B., Y. Zhang: Frobenius manifolds and Virasoro constraints. Sel. math., New ser. **5** (1999), 423–466.
- [Ge] Getzler, E.: Intersection theory on $\bar{\mathcal{M}}_{1,4}$ and elliptic Gromov-Witten invariants. J. Amer. Math. Soc. **10** (1997), 973–998.
- [Gi] Givental, A.B.: Elliptic Gromov-Witten invariants and the generalized mirror conjecture. In: Integrable systems and algebraic geometry. Proceedings of the Taniguchi Symposium 1997 (M.-H. Saito, Y. Shimizu, K. Ueno, eds.). World Scientific, River Edge NJ 1998, pp. 107–155.
- [He1] Hertling, C.: Classifying spaces and moduli spaces for polarized mixed Hodge structures and for Brieskorn lattices. Compositio Math. **116** (1999), 1–37.
- [HM] Hertling, C., Yu. Manin: Weak Frobenius manifolds. Int. Math. Res. Notices **1999**–**6**, 277–286.
- [He3] Hertling, C.: Multiplication on the tangent bundle. First part of the habilitation (also math.AG/9910116).

- [He4] Hertling, C.: Frobenius manifolds and moduli spaces for hypersurface singularities. June 2000. Second part of the habilitation.
- [Hi] Hitchin, N.J.: Frobenius manifolds (notes by D. Calderbank). In: Gauge Theory and symplectic geometry, Montreal 1995, J. Hurtubise and F. Lalonde eds., Kluwer Academic Publishers, Netherlands 1997, 69–112.
- [JM] Jimbo, M., T. Miwa: Monodromy preserving deformations of linear ordinary differential equations with rational coefficients II. *Physica* **2D** (1981), 407–448.
- [JMMS] Jimbo, M., T. Miwa, Y. Mori, M. Sato: Density matrix of impenetrable Bose gas and the fifth Painlevé transcendent. *Physica* **1D** (1980), 80–158.
- [JMU] Jimbo, M., T. Miwa, K. Ueno: Monodromy preserving deformations of linear ordinary differential equations with rational coefficients I. *Physica* **2D** (1981), 306–352.
- [Mal] Malgrange, B.: Sur les déformations isomonodromiques, I, II. In: Séminaire de l'ENS, Mathématique et Physique, 1979–1982, Progress in Mathematics vol. 37, Birkhäuser, Boston 1983, pp. 401–438.
- [Man] Manin, Yu.: Frobenius manifolds, quantum cohomology, and moduli spaces. American Math. Society, Colloquium Publ. v. **47**, 1999.
- [Sab] Sabbah, C.: Déformations isomonodromiques et variétés de Frobenius, une introduction. Centre de Mathématiques, Ecole Polytechnique, U.M.R. 7640 du C.N.R.S., no. 2000-05, 251 pages.
- [SK1] Saito, K.: Primitive forms for a universal unfolding of a function with an isolated critical point, J. Fac. Sci. Univ. Tokyo, Sect. IA Math. **28** (1982), 775–792.
- [SK2] Saito, K.: The zeroes of characteristic function χ_f for the exponents of a hypersurface isolated singular point. In: Algebraic varieties and analytic varieties, Advanced Studies in Pure Math. 1, North-Holland Publ. Company, 1983, pp. 195–217.
- [SK3] Saito, K.: Period mapping associated to a primitive form. Publ. RIMS, Kyoto Univ. **19** (1983), 1231–1264.
- [SM] Saito, M.: On the structure of Brieskorn lattices. Ann. Inst. Fourier Grenoble **39** (1989), 27–72.
- [SchSt] Scherk, J., J.H.M. Steenbrink: On the mixed Hodge structure on the cohomology of the Milnor fibre. Math. Ann. **271** (1985), 641–665.
- [St1] Steenbrink, J.H.M.: Mixed Hodge structure on the vanishing cohomology. In: Real and complex singularities, Oslo 1976, P. Holm (ed.). Alphen aan den Rijn: Sijthoff and Noordhoff 1977, 525–562.
- [St2] Steenbrink, J.H.M.: Intersection form for quasi-homogeneous singularities. Comp. Math. **34** (1977), 211–223.
- [Va] Varchenko, A.N.: The asymptotics of holomorphic forms determine a mixed Hodge structure. Sov. Math. Dokl. **22** (1980), 772–775.

CLAUS HERTLING, MATHEMATISCHES INSTITUT DER UNIVERSITÄT BONN,
 BERINGSTRASSE 1, 53115 BONN, GERMANY
E-mail address: hertling@math.uni-bonn.de